Asymptotic solution for solitary waves in a chain of elastic spheres

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We consider a solitary wave in a chain of contacting, but initially unstressed, particles between which the compressive force F as a function of relative approach x is $F=kx^n$. By ''initially unstressed'' we mean that there is zero contact force between neighboring particles that are infinitely far from the crest of the wave. For a chain of elastic spheres in Hertzian contact, $n=\frac{3}{2}$. In this work, n is treated as ''slightly'' greater than 1, and an asymptotic solution for the solitary wave is developed in terms of the associated small parameter. The solution for the propagating velocity wave is found as a slightly perturbed Gaussian. Comparison with numerics shows that the asymptotic solution is very good even for the fairly large value of $n=\frac{3}{2}$ and is substantially more accurate than the presently available approximate solution given by Nesterenko.

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I. INTRODUCTION

The propagation of compression pulses in granular media has been studied by several authors in recent years (see, e.g., Refs. [1–4], and references therein). One problem of interest in this general area is the propagation of solitary waves in one-dimensional lattices (or chains) of elastic spheres.

The contact interaction between the spheres is well described, at sufficiently small speeds, by Hertz's static solution for the contact of elastic spheres (see, e.g., the detailed experimental study reported in Ref. [4]). In other words, the spheres behave as point masses interacting through massless nonlinear springs whose force F as a function of relative approach x is given by $F=kx^{3/2}$ (see, e.g., Ref. [5]). The constant k is a function of material properties and the radius of the spheres. The $\frac{3}{2}$ power in the contact force law is a geometrical effect; the spring force has no linear part for infinitesimal compression, though the spheres are made of a linearly elastic material. Nesterenko [6] has referred to a chain of such spheres as a ''sonic vacuum.''

In this paper we consider a solitary wave that can propagate down a chain of such balls. Modeling the balls as point masses interacting through Hertzian ''springs'' (i.e., $F=kx^{3/2}$), we numerically examine a case where, as the disturbance propagates down the chain of balls, the solution converges rapidly to a solitary wave. (The existence of this solitary wave has been supported by experiments and numerical simulations, but is not rigorously proven. A closed form solution is currently unavailable.) We then develop an asymptotic description of the tail of the wave, and use it to check the accuracy of the numerical solution far from the wave crest. Finally, we develop a new asymptotic solution, for the traveling wave, that is valid near the wave crest. The new asymptotic solution is significantly more accurate than the currently available approximate solution due to Nesterenko [1].

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II. PRELIMINARIES

Consider a chain of N identical particles of mass m that interact through nonlinear springs that obey the force law $F=k[x]^n$, where x is the distance through which the spring has been compressed and

$$[x]=\begin{cases} x, & x\geqslant 0,\\ 0, & \text{otherwise.}\end{cases} \tag{1}$$

For spheres in Hertzian contact, $n=\frac{3}{2}$. In this paper, we only consider the case $n>1$.

Let the displacement of the ith particle be $\hat{u}_i(t)$, where t is time. Then

$$m\ddot{\hat{u}}_i=k[\hat{u}_{i-1}-\hat{u}_i]^n-k[\hat{u}_i-\hat{u}_{i+1}]^n,\quad i=2,3,\ldots,N-1, \tag{2}$$

where overdots denote time derivatives. For $i=1$, the first term on the right-hand side of Eq. (2) is dropped; while for $i=N$ the second term is dropped. We define $\hat{u}_i(t)=(m/k)^{1/(n-1)}u_i(t)$, to obtain

$$\ddot{u}_i=[u_{i-1}-u_i]^n-[u_i-u_{i+1}]^n,\quad i=2,3,\ldots,N-1, \tag{3}$$

where again the first and second terms on the right-hand side are dropped for $i=1$ and N, respectively.

Let $U:=\{u_1,u_2,\ldots,u_N\}$. It is easy to show by direct substitution that if a function $\tilde{U}(t)$ satisfies Eq. (3), then so does the function $\alpha^{2/(n-1)}\tilde{U}(\alpha t)$, for any positive number α. It follows that if $\tilde{V}(t):=\tilde{U}'(t)$ represents the velocities of the balls in the first solution, then $\alpha^{2(n-1)+1}\tilde{U}'(\alpha t)=\alpha^{(n+1)/(n-1)}\tilde{V}(\alpha t)$ represents the velocities in the second solution.

For example, taking $n=\frac{3}{2}$ (Hertz contact) and α=2, we see that for every solution with velocities $\tilde{V}(t)$, there is another solution with velocities $2^5\tilde{V}(2t)$. For solitary waves in a chain of Hertzian spheres, this means that a disturbance of 2^5 times greater magnitude travels two times as fast.

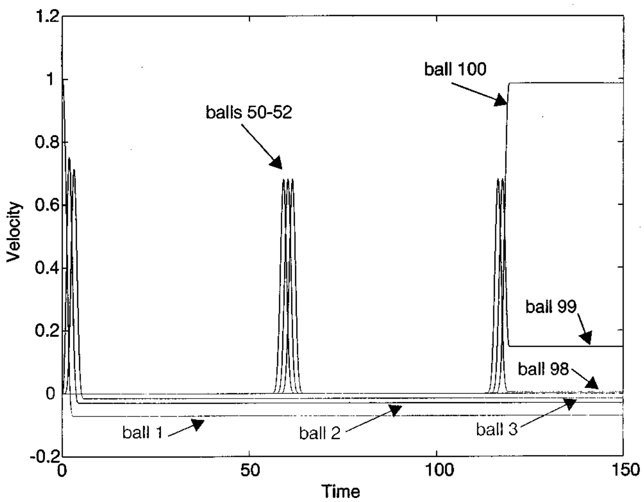

FIG. 1. Velocities of balls as functions of time.

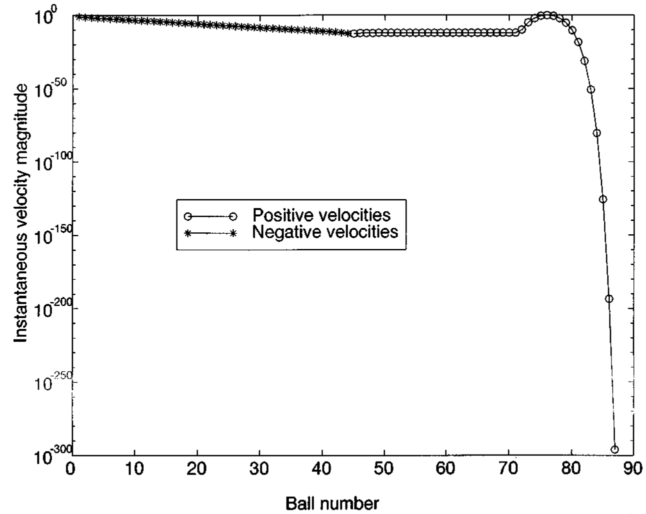

FIG. 2. Snapshot of velocity magnitudes at some intermediate time.

III. A NUMERICAL STUDY

We now numerically study the behavior of the system in Eq. (3) with $n=\frac{3}{2}$, $N=100$, and initial conditions

$$u_i(0)=0, \quad i=1,2,\ldots,N, \quad \dot{u}_1(0)=1,$$

$$\text{and} \quad \dot{u}_i(0)=0, \quad i=2,3,\ldots,N. \tag{4}$$

The aim of this numerical study is twofold: first, to show as carefully as possible (using merely numerics) that there does seem to exist a solitary wave solution for an infinite chain of such balls and second, to demonstrate a numerical method for finding that solution. It is worth examining the numerical solution with some care, since it will be used later to check the accuracy of the asymptotic solution.

The initial conditions used here correspond to a long line of touching, initially stationary and unstressed balls struck at one end by an identical ball. It will be seen below that for these initial conditions it is easy to identify the solitary wave solution for the case where the balls are initially touching but unstressed (i.e., in the limit of zero static contact force at large distances from the wave crest).

The equations of motion with the above initial conditions were solved numerically using MATLAB. It was observed that a disturbance propagates down the chain of balls. Some time after the start of the simulation, ball 1 bounces back with some negative velocity (about -0.071), loses contact with the rest of the chain, and maintains the same velocity thereafter. Some time later, ball 2 bounces back with a *smaller* velocity (about -0.030). Then, ball 3 bounces back with a still smaller velocity, and so on. This situation is seen in Fig. 1, where the velocities of some of the balls are shown as functions of time.

As each ball bounces back with a negative velocity, it carries some kinetic energy away with it; however, the total energy carried away in this manner is bounded, and the system converges rapidly to a limiting situation where the balls appear to bounce back with zero velocity (e.g., see the velocities of balls 50–52 in Fig. 1). In that limiting situation, the disturbance apparently propagates down the chain with no further loss of energy, no further change in shape, and in the form of a solitary wave. For a sufficiently long chain (here, $N=100$) the end effects from the initially impacting end (ball 1) soon become negligible, and the shape of the solitary wave can thus be numerically obtained with great accuracy (more on this later).

The solitary wave is strongly localized, with essentially all the kinetic energy in just a few balls. This may be seen in the figure: the velocity of ball 52 is still very small when the velocity of ball 50 reaches its maximum. Thus, the bulk of the kinetic energy is concentrated within about 5 balls (more on this later).

Eventually, the disturbance reaches the end of the chain. Ball 100 has a velocity somewhat under 1 (about 0.986) when it finally loses contact with ball 99, ball 99 is left with a smaller velocity (about 0.149), ball 98 is left with a still smaller velocity, and so on. This is seen, also, in Fig. 1.

We now examine the numerical solution somewhat more carefully, so as to justify the observations made in the preceding paragraphs. Figures 2 and 3 show snapshots of velocity magnitudes at an instant when the disturbance has propagated about three quarters of the way down the chain of 100 balls.

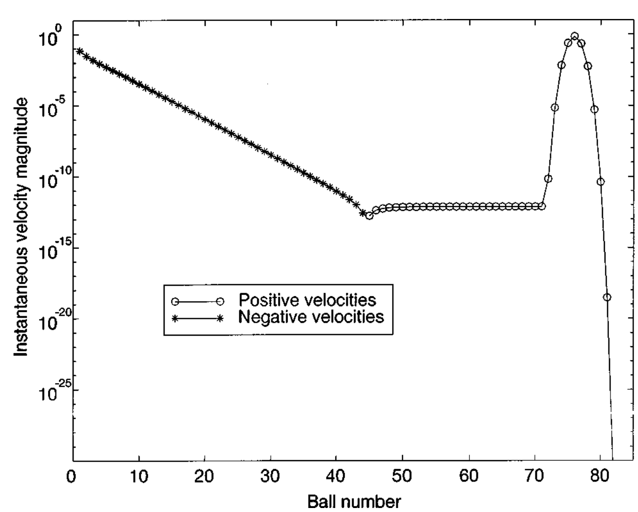

FIG. 3. Enlarged view of a portion of Fig. 2.

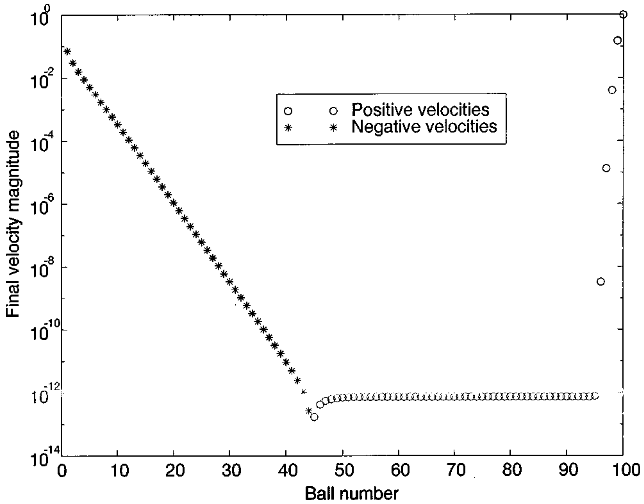

FIG. 4. Final velocities when balls are no more in contact.

Figure 2 shows the degree to which the disturbance is localized: the numerical solution predicts that about 11 balls ahead of the crest of the wave, the speed is smaller by a factor of almost 10^{-300}. Such small numbers from a numerical simulation are usually suspect: however, we discuss in Appendix A a reason why the numerical solution *ahead* of the wave is probably accurate to several significant digits even though its absolute magnitude is so small (in other words, the relative error is small). Also, the match between the numerical solution presented here, and the asymptotic solution presented later, indicates that both solutions are indeed accurate. [As a separate and crude check of numerical integration accuracy, the final kinetic energy of the system, after the disturbance had propagated all the way through the chain of 100 balls, was found to differ from the original kinetic energy (which was $\frac{1}{2}$) by about 10^{-10}.]

Figure 3 shows a portion of Fig. 2. The behavior behind the wave is seen more clearly in this figure. The stars indicate negative velocities, and the circles indicate positive velocities. Examine, first, the roughly linear drop (on a log scale) in the velocities with which successive balls bounce back (balls 1–42). This indicates that the rebound velocity decreases roughly exponentially with ball number, and so the system converges exponentially to a situation where the balls bounce back with zero velocity. Note that the desired numerical integration accuracy specified for MATLAB's solver was 10^{-11}. The numerical solution is unable to capture the zero rebound velocity, and balls 45–70 have final velocities of about 10^{-12}. In Appendix A we discuss a reason why the numerical solution behind the wave is expected to have an accuracy roughly comparable to the ODE solver's numerical integration accuracy, and not be as accurate as the solution ahead of the wave. The wave is seen ahead of ball 70, and it is seen that 3 balls away from the crest of the wave, the velocity is about 10^{-5} times smaller. Thus, the bulk of the kinetic energy is contained in about 5 balls, as estimated by Nesterenko [1].

Figure 4 shows the velocities of the balls when the disturbance has reached the end of the chain. It is seen that the velocities of balls 1–70 are the same as at the earlier time depicted in Fig. 3, since those balls have been moving with constant velocity. It is also seen in the figure that the final kinetic energy is largely concentrated in ball 100, whose velocity is significantly larger than that of ball 99, which is in turn significantly larger than that of ball 98, and so on.

IV. ASYMPTOTIC BEHAVIOR OF THE TAIL

This section provides some analytical support for the foregoing claim that the numerical solution ahead of the wave, as shown in Figs. 2 and 3, is indeed accurate and may be trusted. It also provides an analytical description of how strongly localized the disturbance is in the traveling wave.

We now assume that there is a solitary wave solution for an infinite chain of balls in initial contact, but with no static contact forces far from the wave crest. The displacements of each ball are now described by the same function, just evaluated at different values of the argument, that is,

$$u_i(t)=u_{i+1}(t+b), \quad \text{for some positive constant } b.$$

Arbitrarily picking one particular ball as a reference, we obtain an equation that the traveling wave solution must satisfy

$$\ddot{u}(t)=[u(t+b)-u(t)]^n-[u(t)-u(t-b)]^n,$$

where $u(t)$ is the displacement of the reference ball, overdots denote derivatives with respect to time t and b is a parameter that specifies the speed with which the wave travels down the chain of balls (large b corresponds to small propagation velocity). The function $u(t)$ satisfies the following conditions:

$$\lim_{t\to-\infty} u(t)=0, \quad \lim_{t\to\infty} u(t)=u_\infty, \quad \text{and } \dot{u}(t)>0 \quad \forall t. \tag{5}$$

The numerical solution suggests that the function $v(t)=\dot{u}(t)$ is shaped similar to a symmetric hump. Accordingly, we assume a symmetrical solution. Since the equations are autonomous, the maximum velocity may be chosen to occur at $t=0$, and then $v(t)$ is an even function of time.

By the discussion in Sec. II, b may be scaled by any positive number, and will merely scale the solution $u(t)$ by some corresponding factor depending on n. We pick $b=1$, obtaining

$$\ddot{u}(t)=[u(t+1)-u(t)]^n-[u(t)-u(t-1)]^n. \tag{6}$$

As seen in Fig. 2, the ratio of velocities of successive balls, i.e., $v_{i+1}(t)/v_i(t)$, apparently goes to zero quickly for increasing i and fixed t. We now use this observation to develop an asymptotic description for the tail of the wave.

The portion ahead of the wave in Fig. 2 corresponds to small but rapidly increasing $u(t)$. By the foregoing discussion,

$$u(t+1)\gg u(t)\gg u(t-1)$$

in this regime. Similarly, the portion behind the wave corresponds to $u(t)$ that settles very rapidly to u_∞, and so

$$u_\infty-u(t+1)\ll u_\infty-u(t)\ll u_\infty-u(t-1)$$

in this regime. In the latter regime, defining

$$p(t):=u_\infty-u(t)$$

we have from Eq. (6) (dropping some small terms)

$$\ddot{p}(t) \approx p^n(t-1). \tag{7}$$

We change variables to $p(t)=e^{-s(t)}$ and obtain from Eq. (7)

$$[\dot{s}^2(t)-\ddot{s}(t)]e^{-s(t)} \approx e^{-ns(t-1)}.$$

We assume that $s(t)$ is a fast-increasing function of time, such that $\dot{s}^2 \gg |\ddot{s}|$, and obtain (again dropping some small terms)

$$\dot{s}^2(t)e^{-s(t)} \approx e^{-ns(t-1)}.$$

We try a solution of the form

$$s(t)=Ae^{at}+y(t),$$

where $A>0$ and $|y(t)| \ll e^{at}$, and find (yet again dropping some small terms)

$$a^2A^2e^{2at}e^{-Ae^{at}-y(t)} \approx e^{-nAe^{at-a}-ny(t-1)}.$$

Taking the logarithms of both sides we get the approximate equation

$$-Ae^{at}-y(t)+2at+2\ln(aA)=-ne^{-a}Ae^{at}-ny(t-1),$$

in which the exponentially large terms can balance only if

$$a=\ln(n).$$

For this choice of a, $y(t)$ has a solution of the form C_0t+C_1, where (assuming $n>1$)

$$C_0=\frac{2\ln n}{1-n}, \quad \text{and} \quad C_1=\frac{2}{1-n}\left(\frac{n\ln n}{n-1}+\ln(A\ln n)\right).$$

Note that C_1 is real for $A>0$ only if $n>1$; in that case C_0 is negative. Retaining the old symbols for brevity, we find that

$$p(t) \sim e^{-Ae^{at}-C_0t-C_1}, \tag{8}$$

where the constant A is indeterminate. We should now check *a posteriori* whether any of the simplifying assumptions made earlier are actually violated by the solution. To this end, note that

$$\frac{p(t+1)}{p(t)} \sim e^{(-Ae^{at+a}-C_0t-C_0-C_1)-(-Ae^{at}-C_0t-C_1)}$$
$$=e^{-A(e^a-1)e^{at}-C_0} \ll 1,$$

i.e., $o(1)$ for large t, provided $A>0$, and $a>0$ or $n>1$. It follows that quantities of the form [compare with the small terms dropped from the right-hand side of Eq. (7)]

$$p^n(t-1)+O[p^n(t)]=p^n(t-1)[1+o(1)]=e^{-ns(t-1)+o(1)},$$

because $1+o(1)=e^{o(1)}$. Since the asymptotic solution for the ''large'' quantity s was terminated at $O(1)$ terms, it can be seen that the neglected $o(1)$ terms would not have affected the solution. Checking subsequent steps in a similar way, it is easy to see that only $o(1)$ terms have been neglected, and so the asymptotic description of Eq. (8) is valid.

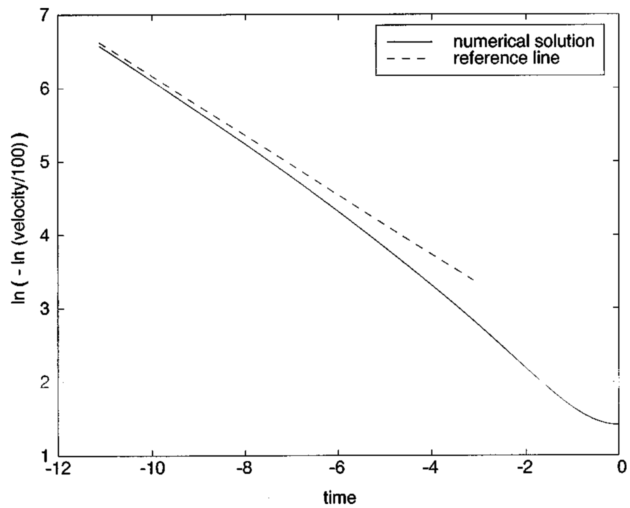

FIG. 5. Comparison between numerical solution and asymptotic prediction far from wave crest.

Differentiating Eq. (8), we find

$$v(t)=\dot{u}(t)=-\dot{p}(t) \sim (aAe^{at}+C_0)e^{-Ae^{at}-C_0t-C_1}$$
$$\sim aAe^{at}e^{-Ae^{at}-C_0t-C_1}.$$

Thus, for any positive constant C_2,

$$\ln[C_2v(t)] \sim \ln C_2+\ln a+\ln A-Ae^{at}+at-C_0t-C_1 \sim -Ae^{at},$$

in which the leading term is independent of C_2 for large t. Taking logarithms again,

$$\ln\{-\ln[C_2v(t)]\} \sim \ln A+at=\ln A+t\ln n \quad \text{as } t\to\infty.$$

Recalling that $v(t)$ is an even function of t, it follows that

$$\ln\{-\ln[C_2v(t)]\} \sim \ln A-at=\ln A-t\ln n \quad \text{as } t\to-\infty.$$

We now compare this asymptotic prediction with our numerical calculations. Note that the previously computed numerical solution [for the initial conditions of Eq. (4)] does not converge to a wave traveling with unit velocity; that is, it does not correspond to a delay of 1 between the motions of successive balls. But, as discussed in Sec. II, it can be (and was) scaled to such a solution. [The delay in the numerical solution itself was estimated using the difference between the instants of maximum velocity (in turn estimated by locally fitting parabolas) for two different balls some distance apart in the chain.]

Ball 50 is far enough from ball 1 for transients to have decayed to numerical integration accuracy, yet far enough from ball 100 for effects from that end to be negligible. We take $v(t)$ to be the scaled velocity of ball 50, obtained from the numerical solution. The instant of maximum velocity is set to $t=0$. Recall that the solution ahead of the wave is considered accurate in Fig. 2; this corresponds to the rising part of the function $v(t)$, i.e., to large negative values of t. Taking $C_2=\frac{1}{100}$ for graphical convenience, $\ln(-\ln[C_2v(t)])$ is shown in Fig. 5 for ''large'' negative t values. A dashed reference line with the predicted slope of

$-a=-\ln n=-\ln(\frac{3}{2})$ is also shown for comparison. It is seen that the slope of the numerically obtained curve appears to be approaching the predicted asymptotic limit. This match may be taken to mean that the asymptotic description of the tail of the wave, as developed above, is correct; at the same time, the numerical solution for the extremely small velocities of balls lying ahead of the wave is also reliable (has small relative error).

Finally, this asymptotic description of the tail shows the degree to which the disturbance is localized. At any instant of time t (fixed), as we examine balls increasingly further away from the crest of the wave, their velocities go to zero as the reciprocal of an extremely fast-growing function: to leading order, this function is the exponential of an exponential.

V. NESTERENKO'S APPROXIMATE SOLUTION

Nesterenko [1] has found an approximate solution for the traveling wave, Eq. (6), which may be presented as follows. First, write $u(t+1)$ as $u(t+b)$, where b is treated essentially as a bookkeeping parameter. Writing a Taylor series expansion about $u(t)$,

$$u(t+b)=u(t)+b\,\frac{du(t)}{dt}+\frac{b^2}{2!}\,\frac{d^2u(t)}{dt^2}+\cdots$$

substituting into Eq. (6) and expanding it, in turn, as a power series in b, we obtain

$$\frac{d^2u(t)}{dt^2}=\frac{3}{2}b^{5/2}\sqrt{\frac{du(t)}{dt}}\,\frac{d^2u(t)}{dt^2}+\frac{1}{8}b^{9/2}\sqrt{\frac{du(t)}{dt}}\,\frac{d^4u(t)}{dt^4}+\frac{1}{8}b^{9/2}\frac{[d^2(t)/dt^2][d^3u(t)/dt^3]}{\sqrt{du(t)/dt}}-\frac{1}{64}b^{9/2}\frac{(d^2u(t)/dt^2)^3}{(du(t)/dt)^{3/2}}+O(b^{13/2}).$$

Dropping terms of $O(b^{13/2})$, and then setting the bookkeeping parameter b back to 1, we obtain

$$\frac{d^2u(t)}{dt^2}=\frac{3}{2}\sqrt{\frac{du(t)}{dt}}\,\frac{d^2u(t)}{dt^2}+\frac{1}{8}\sqrt{\frac{du(t)}{dt}}\,\frac{d^4u(t)}{dt^4}+\frac{1}{8}\,\frac{[d^2u(t)/dt^2][d^3u(t)/dt^3]}{\sqrt{du(t)/dt}}-\frac{1}{64}\,\frac{(d^2u(t)/dt^2)^3}{(du(t)/dt)^{3/2}}. \tag{9}$$

The above approximation rests on the assumption that successive time derivatives of the function $u(t)$ have steadily decreasing magnitudes, i.e., that the width of the velocity wave $v(t)=du(t)/dt$ is large compared to the delay $b=1$.

Equation (9) is satisfied (see Ref. [1]) by

$$v(t)=\frac{du(t)}{dt}=\frac{25}{16}\cos^4\left(\frac{2t}{\sqrt{10}}\right), \tag{10}$$

as may be checked by direct substitution. While this provides a periodic solution, note that taking $v(t)$ to be given by this function for $t\in(-\sqrt{10}\pi/4,\sqrt{10}\pi/4)$, and setting $v(t)\equiv 0$ outside that interval, provides a function $v(t)$ that is three times differentiable as required, satisfies Eq. (9) everywhere [provided $v(t)\equiv 0$ is accepted as a valid solution], and satisfies the basic conditions on the traveling wave solution except for the strict inequality $v(t)>0$ [Eq. (5)].

Thus, Nesterenko's approximation solution is quite a good one, and captures the essential qualitative features of the traveling wave solution. However, it has the disadvantage of being difficult to improve by higher order corrections. In what follows, we develop an asymptotic solution that does not have this disadvantage.

VI. ASYMPTOTIC SOLUTION FOR THE TRAVELING WAVE

The asymptotic behavior of the tail, as developed in Sec. IV, is not useful for obtaining a description of the function $v(t)$ near its crest, i.e., for t comparable to the width of the wave or smaller. In this section we develop an asymptotic solution for the traveling wave using a different approach, and compare our results with the numerical solution presented and validated in previous sections, and with Nesterenko's approximation [Eq. (10)]. The solution developed here fails to capture the qualitative behavior of the function $v(t)$ for sufficiently large t; however, it does decay to zero very fast, and so the absolute error in the solution stays small even though the relative error becomes large for large enough t.

Recall from Sec. II that if $\tilde{V}(t)$ represents the velocities of the balls in one solution to Eq. (3), $\alpha^{(n+1)/(n-1)}\tilde{V}(\alpha t)$ for any $\alpha>0$ also gives a solution. Note that the exponent $(n+1)/(n-1)$ gets very large as n approaches 1, and so we designate

$$\frac{n+1}{n-1}=\frac{1}{\epsilon^2}\quad\text{or}\quad n=\frac{1+\epsilon^2}{1-\epsilon^2}. \tag{11}$$

[As will be clear from the following discussion, the choice of Eq. (11) is somewhat arbitrary. Any choice of the form $n=1+O(\epsilon^2)$, such as $1+\epsilon^2$ or e^{ϵ^2}, can be used in an identical procedure to get a similar solution with the same form but different coefficients. The specific form chosen does not affect the validity of the asymptotic series, but may affect the rapidity with which the series converges (if it converges at all) for any fixed value of ϵ.] Thus, $n=\frac{3}{2}$ when $\epsilon=1/\sqrt{5}$. We now attempt to approximately solve Eq. (6) for small ϵ.

In Eq. (6), for small ϵ the right-hand side gets close to $u(t+1)-2u(t)+u(t-1)$, which is a finite difference version of the left-hand side. We therefore suspect that the hump in the wave must get wider and wider as ϵ goes to zero. Some numerical solutions (not presented here) suggest that for small ϵ, the width of the hump in the solitary wave is roughly proportional to $1/\epsilon^2$.

Accordingly, we introduce the scaled or slow time $\tau=\epsilon t$. Equation (6) becomes

$$\epsilon^2 \frac{d^2u(\tau)}{d\tau^2} = [u(\tau+\epsilon) - u(\tau)]^{(1+\epsilon^2)/(1-\epsilon^2)} - [u(\tau) - u(\tau-\epsilon)]^{(1+\epsilon^2)/(1-\epsilon^2)}. \quad (12)$$

Expanding Eq. (12) as a series in terms of ϵ (using the symbolic algebra package MAPLE), and substituting $V(\tau) := du(\tau)/d\tau$, gives to lowest order

$$\epsilon^4 \left(\frac{dV}{d\tau} + \frac{1}{24}\frac{d^3V}{d\tau^3} + \frac{dV}{d\tau}\ln V + \frac{dV}{d\tau}\ln \epsilon \right) + O(\epsilon^6) = 0, \quad (13)$$

where, as is common is expansions simultaneously involving powers and logarithms of a small parameter, we treat $\ln \epsilon$ essentially as an $O(1)$ quantity compared to quantities such as ϵ, ϵ^2, and so on (see, e.g., Ref. [7]).

Retaining and integrating the leading order term from Eq. (13), we obtain

$$\frac{1}{24}\frac{d^2V}{d\tau^2} + V\ln V + V\ln \epsilon = B_0, \quad \text{an arbitrary constant.}$$

Since $V \to 0$ as $\tau \to \infty$, $B_0 = 0$. Multiplying by $dV/d\tau$ and integrating again, we obtain

$$\frac{1}{48}\left(\frac{dV}{d\tau}\right)^2 + \frac{V^2}{2}\ln V - \frac{1}{4}V^2 + \frac{\ln \epsilon}{2}V^2 = B_1, \quad \text{an arbitrary constant.}$$

Since $V \to 0$ as $\tau \to \infty$, $B_1 = 0$ as well. Dividing by $V^2/2$, we obtain

$$\frac{1}{24}\frac{1}{V^2}\left(\frac{dV}{d\tau}\right)^2 + \ln V - \frac{1}{2} + \ln \epsilon = 0.$$

Letting $Y := \ln V$, we obtain

$$\frac{1}{24}\left(\frac{dY}{d\tau}\right)^2 + Y - \frac{1}{2} + \ln \epsilon = 0,$$

which may be integrated to give

$$\frac{1}{2} - \ln \epsilon - Y = (\sqrt{6}\tau + B_2)^2,$$

where B_2 is an arbitrary constant. Since, by choice of the origin $t = 0$ at the instant of maximum velocity, V and hence Y is an even function of τ, B_2 must be zero. Thus, we finally obtain the leading or *first order* approximation to V as

$$\exp\left(\frac{1}{2} - \ln\epsilon - 6\tau^2\right).$$

Guided by the solution so far, as well as the preceding numerical results, we now assume a solution of the form

$$V(\tau) = \exp\left(\frac{1}{2} - \ln \epsilon - 6\tau^2 + \epsilon^2 V_2(\tau) + \epsilon^4 V_4(\tau) + \cdots\right).$$

In other words, since the leading order solution is a Gaussian function, we seek a solution in the form of a perturbed Gaussian. Substituting this form into Eq. (12) and expanding gives, at $O(\epsilon^6)$,

$$\frac{1}{12}\frac{d^3V_2}{d\tau^3} - 3\tau\frac{d^2V_2}{d\tau^2} + 24\tau^2\frac{dV_2}{d\tau} - 24\tau V_2 = -18\tau + 144\tau^3 - \frac{864}{5}\tau^5. \quad (14)$$

The complementary solution to the linear ODE, Eq. (14), contains the fast-growing functions

$$\exp(r\tau^2) \quad \text{and} \quad \tau\int_0^\tau \exp(rx^2)\,dx, \quad r = 6 \text{ and } 12.$$

However, to develop a solution consistent to this order, we do not need the general solution to Eq. (14). Instead, we note that Eq. (14) has a particular solution in the form of the more slowly growing even polynomial

$$V_2 = -\frac{1}{20} + \frac{12}{5}\tau^2 - \frac{12}{5}\tau^4. \quad (15)$$

Since this polynomial grows more slowly than the complementary function, dropping the contribution from the latter is in the spirit of dropping secular terms.

At the next order in ϵ^2, the equation for V_4, not reproduced here, has the same left-hand side (or differential operator) as Eq. (14), a different polynomial on the right-hand side, and again permits an even polynomial solution

$$V_4 = -\frac{211}{2100} + \frac{6}{25}\tau^2 + \frac{48}{175}\tau^4 - \frac{96}{175}\tau^6. \quad (16)$$

In this way each successive order in the expansion provides even polynomials of increasingly higher order. We terminate the expansion at second order here; however, two more terms are provided in Appendix B.

Note that

$$V(\tau) = \frac{du}{d\tau} = \frac{1}{\epsilon}\frac{du}{dt},$$

and so we obtain the approximation to three terms, or third order,

$$v(t) = \frac{du}{dt} \approx \exp\left(\frac{1}{2} - 6\tau^2 + \epsilon^2 V_2(\tau) + \epsilon^4 V_4(\tau)\right), \quad (17)$$

where $\tau = \epsilon t$, and V_2 and V_4 are given in Eqs. (15) and (16). Naturally, as discussed earlier, this solution may be scaled up or down, to obtain a family of faster or slower solitary wave solutions. We can now compare the numerical solution, Nesterenko's approximation, and the asymptotic solution developed above.

Figure 6 shows the region near the crest of the wave. One hump from Nesterenko's periodic solution, Eq. (10), is shown plotted for comparison. The first order solution developed here is fairly good; the second order solution is better; and the third order solution (i.e., up to V_4) is nearly indis-

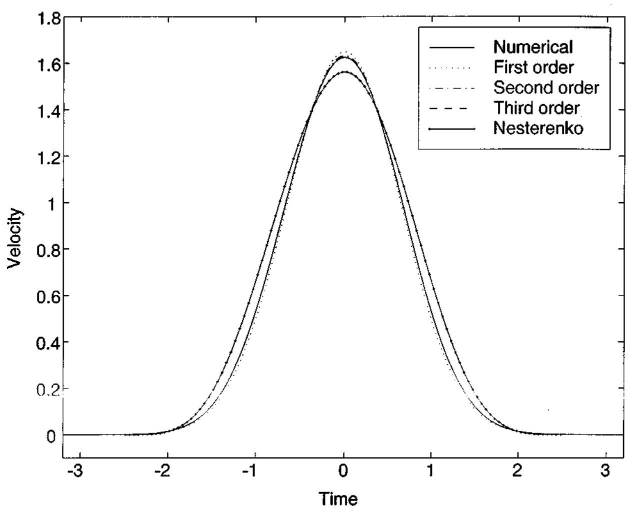

FIG. 6. Comparison between numerical solution, Nesterenko's approximation, and Eq. (15).

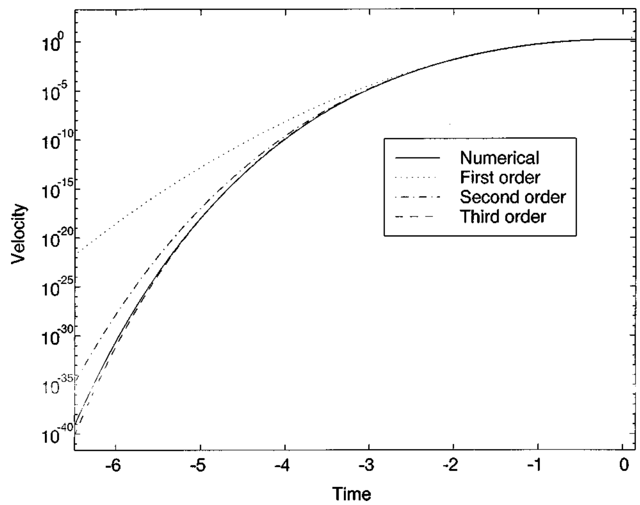

FIG. 8. Behavior far from the crest.

tinguishable from the numerical solution, both in Fig. 6, as well as in Fig. 7 which shows an enlargement of a portion of Fig. 6.

Finally, note that the asymptotic solution developed here formally breaks down when τ gets sufficiently large, e.g., when $\tau = O(1/\sqrt{\epsilon})$. It is therefore interesting to see if the solution is any good for larger values of τ.

A feature of the true solution is that it decays to zero very rapidly at large distances from the crest of the wave. Thus, so long as the asymptotic solution is truncated at an order where the coefficient of the highest power in τ is *negative*, the correct qualitative behavior will be predicted for all τ, and the absolute error in $v(t)$ will be uniformly small, though the relative error must eventually get large. As mentioned in Appendix B, this means that the expansion may be truncated at V_2, V_4, or V_8, but not at V_6.

The relative error in the approximation, at large distances from the crest of the wave, can be seen by comparing the solutions on a semilog plot. As seen in Fig. 8, the first three orders of approximation give increasingly better approximations away from the crest, and the solution up to V_4, i.e., Eq. (17), has small relative error until the solution has decayed to many orders of magnitude smaller than its maximum value.

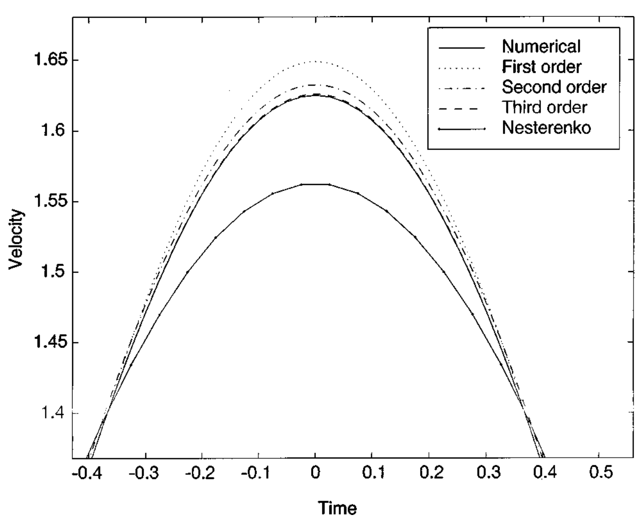

FIG. 7. A portion of Fig. 6.

VII. CONCLUSIONS

The problem of a solitary wave in a one-dimensional lattice with power law interaction has been studied in this paper. Attention has been restricted to the case where the power law index n is greater than but close to unity; and where the static contact force is zero at large distances from the wave crest. The index of greatest practical interest is $n = \frac{3}{2}$, corresponding to Hertzian contact between elastic spheres.

A careful numerical solution has been used to motivate and guide the analysis. A separate asymptotic description of the tail of the wave has been used to validate the numerical solution. Finally, an asymptotic solution has been developed for the full solitary wave. This solution is formally valid at distances from the wave crest comparable to the width of the wave. However, the strongly localized nature of the wave ensures that the absolute error in the approximate solution is small everywhere. For the case of primary interest, i.e., $n = \frac{3}{2}$, the solution is substantially more accurate than the currently available approximate solution. The sort of asymptotic expansion used here (perturbed Gaussian) may prove useful in other types of systems as well, such as systems with interaction laws that are slightly perturbed power laws.

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APPENDIX A: ON CATASTROPHIC CANCELLATION

The problem of catastrophic cancellation is well known (see, e.g., Ref. [8]), but is briefly described here for completeness. Consider the problem of summing a sequence of numbers using finite precision arithmetic.

First consider summing many numbers of the same sign, from a series that converges fast. For example, consider summing the series

$$\epsilon^{35}=1+35+\frac{35^2}{2!}+\frac{35^3}{3!}+\cdots .$$

Carrying out the computation using double precision (in MATLAB), and taking 180 terms, yields a sum that matches MATLAB's value to 15 significant digits, i.e., the relative error is very small.

Now consider summing many numbers, both positive and negative, such that the exact sum is a very small number. For example, consider

$$e^{-35}=1-35+\frac{35^2}{2!}-\frac{35^3}{3!}+\cdots$$

(this well known example is mentioned in Ref. [8]). Now taking 180 terms yields -1.47×10^{-2} while MATLAB's value is about 6.3×10^{-16}. Thus, the absolute error in the computation is on the order of 10^{-2}, while the relative error is very large indeed (the computed sum is correct to zero significant digits). A crude explanation for this is that the largest term in the series is about $35^{35}/35!\approx10^{14}$. With about 16 digits of accuracy, that term can only be calculated to an absolute accuracy of about 10^{-2}, and the summing procedure never recovers from this loss of accuracy. Thus, the absolute error is roughly determined by the absolute precision with which the largest term is calculated, and quite independent of the final (correct) sum.

In the light of this discussion, consider the numerical solution presented in Sec. III. The numerical ODE solver essentially computes a series of increments to the state vector as it marches forward in time. Ahead of the wave, the velocities of the balls start at zero; and the increments in them are all positive, i.e., of the same sign. Moreover, since the disturbance is strongly localized, the velocity increments with successive time steps increase very rapidly, which is similar to summing a rapidly converging series in reverse. Thus, the relative error in the sum is small, and the numerical result is accurate. However, *behind* the wave, the velocity increments are negative, and begin to reduce the velocity from its maximum value (comparable to 1). Thus, the absolute error accumulated by the integrator stays at about 10^{-12} (the integration tolerance specified in the numerical solution was 10^{-11}) even as the true solution decays to much smaller values. For this reason, the numerical solution ahead of the wave is expected to be significantly more accurate than that behind the wave.

APPENDIX B: HIGHER ORDER CORRECTIONS

Carrying through the procedure described in Sec. VI, we obtain (using MAPLE)

$$V_6=-\frac{313}{7000}+\frac{1632}{875}\tau^2-\frac{6072}{875}\tau^4+\frac{576}{4375}\tau^6+\frac{288}{1225}\tau^8$$

and

$$V_8=-\frac{199\,791}{2\,695\,000}+\frac{629\,814}{336\,875}\tau^2-\frac{2\,806\,752}{336\,875}\tau^4+\frac{11\,121\,696}{1\,684\,375}\tau^6+\frac{5\,642\,496}{11\,790\,625}\tau^8-\frac{1536}{48\,125}\tau^{10}.$$

The expressions get increasingly length for the higher order corrections. As shown earlier, truncating at $O(\tau^4)$ works very well for $n=\frac{3}{2}$ or $\epsilon^2=\frac{1}{5}$. If higher order corrections are used, the expansion should be truncated at an order j such that the function V_{2j} (such as V_6 or V_8, as given above) has a negative coefficient on the highest power of τ. This is because the asymptotic expansion developed here is valid only for $\tau=O(1)$, and is just being used for all τ because $V(\tau)$ decays rapidly to zero. A positive coefficient on the highest power would eventually cause the solution to start growing, for large enough τ, rendering the solution useless for large τ. Thus, V_4 and V_8 are reasonable places to terminate the expansion, but V_6 is not.

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